Lecture 17: October 30

Proof of the technical result. We are in the process of proving the one-dimensional version of the Cattani-Deligne-Kaplan theorem about the locus of Hodge classes. We first need to finish up the proof of the following technical result.

Proposition. *Suppose that $z_n \in \tilde{\mathbb{H}}$ is a sequence of points with bounded imaginary parts, such that $t_n = e^{z_n} \to 0$. Also suppose that $v_n \in V_{\mathbb{Z}}$ is a sequence of integral classes with $h(v_n, v_n) \leq K$, such that $v_n \in F^0_{\Phi(z_n)}$ for every $n \in \mathbb{N}$. Then after passing to a subsequence, v_n is constant, and the constant value belongs to $F^0_{\Psi(0)} \cap \ker R$.*

We already proved that, after passing to a subsequence, the $E_0(H)$-component of v_n is constant, and $Rv_n = 0$. It remains to show that the sequence $v_n \in V_{\mathbb{Z}}$ can only take finitely many values; or, what amounts to the same thing, that a subsequence is constant.

Step 4. We prove that the sequence v_n can take only finitely many values, and that every constant subsequence lies in $F^0_{\Psi(0)}$. The idea is to bound the Hodge norm of v_n with respect to a *fixed* Hodge structure on V. Recall that the two holomorphic mappings $\Phi\colon \tilde{\mathbb{H}} \to D$ and $\Psi\colon \Delta \to \check{D}$ are related by the formula $\Psi(e^z) = e^{-zR}\Phi(z)$. Since $Rv_n = 0$, we have

$$v_n = e^{-z_n R} v_n \in e^{-z_n R} F^0_{\Phi(z_n)} = F^0_{\Psi(t_n)}.$$

Since Ψ is holomorphic, the subspaces on the righ-hand side converge to $F^0_{\Psi(0)}$ at a rate of $|t_n|$. We can therefore decompose

$$v_n = v_n' + v_n''$$

with $v_n' \in F^0_{\Psi(0)}$ and v_n'' of size bounded by a constant multiple of $|t_n|$. Recall from the exercises in Lecture 13 that we can find $z \in \tilde{\mathbb{H}}$ with $\operatorname{Re} z \ll 0$, such that $e^{zR}\Psi(0) \in D$. The above decomposition for v_n gives

$$v_n = e^{zR} v_n' + e^{zR} v_n'';$$

here $e^{zR} v_n' \in e^{zR} F^0_{\Psi(0)}$, and the Hodge norm of $e^{zR} v_n''$ in the Hodge structure $e^{zR}\Psi(0)$ is bounded by a constant multiple of $|t_n|$. By looking at the Hodge decomposition of $v_n \in V_{\mathbb{Z}}$ in the Hodge structure $e^{zR}\Psi(0)$, and using the fact that $h(v_n, v_n) \leq K$, one deduces from this relation that

$$\|v_n\|^2_{e^{zR}\Psi(0)} \leq K + 4C|t_n|^2.$$

In particular, the Hodge norm of v_n is bounded, and since $v_n \in V_{\mathbb{Z}}$, it follows that the sequence v_n can take only finitely many distinct values. Moreover, since $v_n \in F^0_{\Psi(t_n)}$, any value that appears infinitely many times must belong to

$$\lim_{n \to \infty} F^0_{\Psi(t_n)} = F^0_{\Psi(0)}.$$

This completes the proof of the technical result.

The locus of Hodge classes is algebraic. The technical result in Proposition 16.5 is all that we need to prove that the locus of Hodge classes $\operatorname{Hdg}_K(\mathscr{V})$ is algebraic. Last time, we reduced the problem to the case of a polarized variation of $\mathbb{Z}$-Hodge structure $\mathscr{V}$ on a quasi-projective curve, with unipotent local monodromy around each point of $\bar{X} \setminus X$. We also said that, by Chow's theorem, it is enough to construct an extension of $\operatorname{Hdg}_K(\mathscr{V})$ that is finite and proper over $\bar{X}$.

Let us first do this in the local setting where $X = \Delta^*$ and $\bar{X} = \Delta$. Recall that the étalé space of the local system $\mathscr{V}_{\mathbb{Z}}$ is the image of the holomorphic mapping

$$\tilde{\mathbb{H}} \times V_{\mathbb{Z}} \to \Delta^* \times V, \quad (z, v) \mapsto \left(e^z, e^{-zR} v\right). \tag{17.1}$$

Here $\tilde{\mathbb{V}} \cong \Delta \times V$ is the trivialization of the canonical extension, and $\mathbb{V} \cong \Delta^* \times V$ the induced trivialization of $\mathscr{V}$. The locus of Hodge classes $\operatorname{Hdg}(\mathscr{V})$ is the intersection of $\operatorname{\acute{E}t}(\mathscr{V}_{\mathbb{Z}})$ with the subbundle $F^0\mathbb{V}$. In our trivialization, the fiber of the Hodge bundle $F^0\mathbb{V}$ at a point $t \in \Delta^*$ is exactly the subspace $F^0_{\Psi(t)}$. Since we know from Theorem 9.1 that $\Psi \colon \delta \to \check{D}$ is holomorphic, we actually have a subbundle $F^0\tilde{\mathbb{V}}$, whose fiber over the origin in $F^0_{\Psi(0)}$.

Now Proposition 16.5 suggests how to construct an extension of $\operatorname{Hdg}_K(\mathscr{V})$ to an object over Δ. First, it is easy to see that the irreducible components of the image of (17.1) are of two kinds: (1) If $v \in V_{\mathbb{Z}}$ satisfies $Rv = 0$, then the corresponding component of the image is a copy of Δ^*, consisting of all points (t, v) with $t \in \Delta^*$. The closure of such a component also contains the point $(0, v)$. (2) If $v \in V_{\mathbb{Z}}$ satisfies $Rv \neq 0$, then the corresponding component of the image is closed (and isomorphic to $\tilde{\mathbb{H}}$). The closure of $\operatorname{\acute{E}t}(\mathscr{V}_{\mathbb{Z}})$ inside $\tilde{\mathbb{V}} \cong \Delta \times V$ is therefore still a closed analytic subset. If we intersect it with the subbundle $F^0\tilde{\mathbb{V}}$, we get another closed analytic subset, which agrees with the locus of Hodge classes over Δ^*. The points that get added are of the form $(0, v)$, where $v \in V_{\mathbb{Z}}$ satisfies $Rv = 0$ and $v \in F^0_{\Psi(0)}$. You will notice that these are exactly the sort of points that can appear as limits of a sequence of Hodge classes in Proposition 16.5.

We can globalize this construction as follows. The closure of $\operatorname{\acute{E}t}(\mathscr{V}_{\mathbb{Z}})$ inside the vector bundle $\tilde{\mathbb{V}}$ is an analytic subset, and the intersection

$$\widetilde{\operatorname{Hdg}}(\mathscr{V}) = \overline{\operatorname{\acute{E}t}(\mathscr{V}_{\mathbb{Z}})} \cap F^0\tilde{\mathbb{V}} \subseteq \tilde{\mathbb{V}}$$

is therefore a closed analytic subset that extends $\operatorname{Hdg}(\mathscr{V})$. Moreover,

$$\widetilde{\operatorname{Hdg}}_K(\mathscr{V}) = \overline{\operatorname{\acute{E}t}_K(\mathscr{V}_{\mathbb{Z}})} \cap F^0\tilde{\mathbb{V}} \subseteq \tilde{\mathbb{V}}$$

is a union of connected components, corresponding to classes whose self-intersection number is bounded by K, and extends $\operatorname{Hdg}_K(\mathscr{V})$. Both of these live over $\bar{X}$.

Proposition 17.2. *The projection $\widetilde{\operatorname{Hdg}}_K(\mathscr{V}) \to \bar{X}$ is proper with finite fibers.*

Proof. I will only prove properness. This is a local problem, and so it suffices to consider the case where $X = \Delta^*$ and $\bar{X} = \Delta$. Take a sequence of points $(t_n, v_n) \in \operatorname{\acute{E}t}(\mathscr{V}_{\mathbb{Z}})$ with $t_n \to 0$. Properness is the statement that a subsequence converges to a limit in $\widetilde{\operatorname{Hdg}}_K(\mathscr{V})$. But Proposition 16.5 says that, after passing to a subsequence, $v_n = v$ is constant and belongs to $\ker R \cap F^0_{\Psi(0)}$. Therefore $(t_n, v_n) \to (0, v)$, which belongs to $\widetilde{\operatorname{Hdg}}_K(\mathscr{V})$ by construction. □

Since $\bar{X}$ is projective, Chow's theorem implies that $\widetilde{\operatorname{Hdg}}_K(\mathscr{V})$ is also projective; it follows that $\operatorname{Hdg}_K(\mathscr{V})$ is a quasi-projective algebraic variety.

Note. Not every point in $\widetilde{\operatorname{Hdg}}_K(\mathscr{V})$ is the limit of a sequence of Hodge classes. A typical example are vanishing cycles, for example in a one-parameter degeneration of a family of smooth hypersurfaces in $\mathbb{P}^3$ to a surface with an ordinary double point. Each vanishing cycle in a 2-sphere, whose class is generally not a Hodge class, but which becomes a Hodge class "in the limit". This suggest calling the points in $\widetilde{\operatorname{Hdg}}_K(\mathscr{V})$ "limit Hodge classes". So what Cattani, Deligne, and Kaplan really prove is that the locus of limit Hodge classses is a projective algebraic variety.

Schmid's results and Hodge modules. This seems like a good time to start introducing Hodge modules, in the case of the disk. The general idea is that from a polarized variation of Hodge structure on the punctured disk Δ^*, we would like to construct a "Hodge module" on the disk Δ that extends the variation of Hodge structure in a suitable sense. (More generally, given a polarized variation of Hodge structure on a smooth quasi-projective curve X, we would like to have a Hodge module on a projective compactification $\bar{X}$, because it is generally better to work over projective varieties.) Unless the variation of Hodge structure happens to extend to Δ, this object is going to have some kind of singularity at the origin. Schmid's results are going to suggest how this should look like.

Let me start with a brief summary. Our variation of Hodge structure consists of a vector bundle $\mathscr{V}$ with a connection $\nabla\colon \mathscr{V} \to \Omega^1_{\Delta^*} \otimes_{\mathscr{O}_{\Delta^*}} \mathscr{V}$, a flat hermitian pairing $h_{\mathscr{V}}\colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}^\infty_{\Delta^*}$, and a family of subbundles $F^p\mathscr{V}$. We will see that the pair $(\mathscr{V}, \nabla)$ naturally extends to a $\mathscr{D}_\Delta$-module $\mathcal{M}$, where $\mathscr{D}_\Delta$ is the sheaf of differential operators on Δ. The polarization extends to a pairing $h_{\mathcal{M}}\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathrm{Db}_\Delta$ with values in the sheaf of distributions on Δ. Lastly, the Hodge bundles extend to a filtration $F_\bullet\mathcal{M}$ by coherent $\mathscr{O}_\Delta$-modules that is compatible with the action by differential operators. In each of these three cases, the "singularity" of the variation of Hodge structure requires working in a larger class of objects: $\mathscr{D}$-modules instead of vector bundles with connection, distributions instead of smooth functions, and coherent sheaves instead of vector bundles.

Extending the vector bundle with connection. We now take up each of the three elements, starting from the vector bundle $\mathscr{V}$ and the connection ∇. Recall from Lecture 8 that we have a family of canonical extensions, which are holomorphic vector bundles on Δ that extend $\mathscr{V}$. For $\alpha \in \mathbb{R}$, let $\tilde{\mathscr{V}}^\alpha$ be the canonical extension for the interval $[\alpha, \alpha+1)$; recall that this means that the residue $\mathrm{Res}_0(\nabla)$ of the logarithmic connection

$$\nabla\colon \tilde{\mathscr{V}}^\alpha \to \Omega^1_\Delta(\log 0) \otimes_{\mathscr{O}_\Delta} \tilde{\mathscr{V}}^\alpha$$

has its eigenvalues in the interval $[\alpha, \alpha+1)$. Similarly, $\tilde{\mathscr{V}}^{>\alpha}$ means the canonical extension for the interval $(\alpha, \alpha+1]$. Each canonical extension is a subsheaf of $j_*\mathscr{V}$, where $j\colon \Delta^* \hookrightarrow \Delta$ is the open embedding.

The following discussion will be clearer if we briefly recall the construction of $\tilde{\mathscr{V}}^\alpha$ from Lecture 8. Let V be the space of flat section of $\exp^* \mathscr{V}$, where $\exp\colon \tilde{\mathbb{H}} \to \Delta^*$ is the universal covering by the half space $\tilde{\mathbb{H}} = \{\, z \in \mathbb{C} \mid \mathrm{Re}\, z < 0 \,\}$. Write the monodromy transformation $T \in \mathrm{End}(V)$ in the form

$$T = e^{2\pi i R} = e^{2\pi i R_S} e^{2\pi i R_N},$$

where R_N is nilpotent, R_S is semisimple with eigenvalues in the interval $[\alpha, \alpha+1)$, and the two operators commute. The space of flat sections gives us a trivialization $\mathscr{O}_{\tilde{\mathbb{H}}} \otimes_{\mathbb{C}} V \cong \exp^* \mathscr{V}$, and for each $v \in V$, the holomorphic section

$$\tilde{s}_v(z) = \bigl(e^{zR} v\bigr)(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} (R^j v)(z)$$

of the trivial bundle descends to a holomorphic section $s_v \in H^0(\Delta^*, \mathscr{V})$. We constructed $\tilde{\mathscr{V}}^\alpha$ by taking the trivial bundle $\mathscr{O}_\Delta \otimes_{\mathbb{C}} V$, and mapping it into $j_*\mathscr{V}$ by sending $1 \otimes v$ to the section $s_v \in H^0(\Delta, j_*\mathscr{V})$.

The construction shows how the different $\tilde{\mathscr{V}}^\alpha$ are related. If we replace α by $\alpha+1$ in the construction, then R_S changes to $R_S + \mathrm{id}$, and $\tilde{s}_v$ and $s_v(t)$ get multiplied by $t = e^z$. Similarly, if we replace α by $\alpha - 1$, then $s_v(t)$ gets multiplied by $t^{-1} = e^{-z}$. As subsheaves of $j_*\mathscr{V}$, we therefore have $\tilde{\mathscr{V}}^{\alpha+1} = t\tilde{\mathscr{V}}^\alpha$ and $\tilde{\mathscr{V}}^{\alpha-1} = t^{-1}\tilde{\mathscr{V}}^\alpha$. In particular, $\tilde{\mathscr{V}}^{\alpha+1} \subseteq \tilde{\mathscr{V}}^\alpha$. More generally, we have the following lemma.

Lemma 17.3. *If $\alpha \leq \beta$, then $\tilde{\mathscr{V}}^\beta \subseteq \tilde{\mathscr{V}}^\alpha$.*

Proof. Let $\lambda_1, \dots, \lambda_r \in [\alpha, \alpha+1)$ be the distinct eigenvalues of R_S. Since $T = e^{2\pi i R}$ is independent of the choice of interval, the eigenvalues of the residue on $\tilde{\mathscr{V}}^\beta$ must be of the form $\lambda_1+a_1, \dots, \lambda_k+a_k \in [\beta, \beta+1)$ with nonnegative integers $a_1, \dots, a_k \in \mathbb{N}$. Given any $v \in V$, we decompose $v = v_1 + \cdots + v_r$, where $v_j \in E_{\lambda_j}(R_S)$. In the construction of $\tilde{\mathscr{V}}^\alpha$, the section corresponding to $v \in V$ is then

$$s_v(t) = s_{v_1}(t) + \cdots + s_{v_r}(t).$$

In the construction of $\tilde{\mathscr{V}}^\beta$, the section corresponding to $v \in V$ is

$$t^{a_1} s_{v_1}(t) + \cdots + t^{a_r} s_{v_r}(t).$$

This is a linear combination of sections of $\tilde{\mathscr{V}}^\alpha$, with holomorphic functions as coefficients, and so $\tilde{\mathscr{V}}^\beta \subseteq \tilde{\mathscr{V}}^\alpha$. □

Exercise 17.1. Show in a similar manner that $\tilde{\mathscr{V}}^{>\alpha} = \tilde{\mathscr{V}}^{\alpha+\varepsilon}$ for $\varepsilon > 0$ sufficiently small.

The canonical extensions depend on a choice of interval, but the sheaf

$$\tilde{\mathscr{V}} = \bigcup_{\alpha \in \mathbb{R}} \tilde{\mathscr{V}}^\alpha \subseteq j_* \mathscr{V}$$

is independent of any choices. It is called *Deligne's canonical meromorphic extension* of the pair $(\mathscr{V}, \nabla)$. Clearly, $\tilde{\mathscr{V}}$ is a sheaf of $\mathscr{O}_\Delta$-modules that agrees with $\mathscr{V}$ outside the origin; note that $\tilde{\mathscr{V}}$ is typically not coherent over $\mathscr{O}_\Delta$.

Example 17.4. If $\mathscr{V} = \mathscr{O}_{\Delta^*}$, with the trivial connection $d\colon \mathscr{O}_{\Delta^*} \to \Omega^1_{\Delta^*}$, then $\tilde{\mathscr{V}}^0 = \mathscr{O}_\Delta$, and more generally $\tilde{\mathscr{V}}^\ell = t^\ell \mathscr{O}_\Delta$ for every $\ell \in \mathbb{Z}$. In this case, $\tilde{\mathscr{V}}$ is the sheaf of holomorphic functions on Δ^* with poles of arbitrary order at the origin; this is clearly not coherent as an $\mathscr{O}_\Delta$-module.

The logarithmic connection on each $\tilde{\mathscr{V}}^\alpha$ gives $\tilde{\mathscr{V}}$ the structure of a left module over $\mathscr{D}_\Delta$, the sheaf of linear differential operators of finite order. This is a very concrete object in this case, and you don't need to know anything about $\mathscr{D}$-modules to understand what is going on. We have $\mathscr{D}_\Delta = \mathscr{O}_\Delta\langle \partial_t \rangle$, where $\partial_t = \frac{\partial}{\partial t}$ is the derivative operator with respect to the variable t. Note that t and ∂_t do not commute; instead, they satisfy the relation

$$[\partial_t, t] = \partial_t \cdot t - t \cdot \partial_t = 1.$$

More generally, we have $[\partial_t, f] = \frac{\partial f}{\partial t}$ for any $f \in \mathscr{O}_\Delta$. If s is any local section of $\tilde{\mathscr{V}}^\alpha$, we define the action by ∂_t as

$$\partial_t \cdot s = \nabla_{\partial_t} s \in \frac{1}{t} \tilde{\mathscr{V}}^\alpha = \tilde{\mathscr{V}}^{\alpha-1}.$$

The Leibniz rule for the connection reads

$$\partial_t \cdot (fs) = \nabla_{\partial_t}(fs) = \frac{\partial f}{\partial t} s + f \nabla_{\partial_t}(s) = \frac{\partial f}{\partial t} s + f \partial_t \cdot s,$$

and so left multiplication by ∂_t is compatible with the relation $[\partial_t, f] = \frac{\partial f}{\partial t}$. This means that $\tilde{\mathscr{V}}$ is indeed a left $\mathscr{D}_\Delta$-module. Let me emphasize again that

$$t \cdot \tilde{\mathscr{V}}^\alpha = \tilde{\mathscr{V}}^{\alpha+1} \quad \text{and} \quad \partial_t \cdot \tilde{\mathscr{V}}^\alpha \subseteq \tilde{\mathscr{V}}^{\alpha-1},$$

all viewed as subsheaves of $j_* \mathscr{V}$.

Lemma 17.5. *As a left $\mathscr{D}_\Delta$-module, $\tilde{\mathscr{V}}$ is coherent.*

Proof. More precisely, we will show that $\tilde{\mathscr{V}} = \mathscr{D}_\Delta \cdot \tilde{\mathscr{V}}^{-1}$. Since $\tilde{\mathscr{V}}^{-1}$ is a coherent $\mathscr{O}_\Delta$-module – in fact, even locally free – it follows that $\tilde{\mathscr{V}}$ is a coherent $\mathscr{D}_\Delta$-module. In view of how we defined $\tilde{\mathscr{V}}$, it suffices to prove that $\tilde{\mathscr{V}}^\alpha = \partial_t \cdot \tilde{\mathscr{V}}^{\alpha+1}$ as long as $\alpha \leq -2$. Consider the composition

$$\tilde{\mathscr{V}}^\alpha \xrightarrow{\;t\;} \tilde{\mathscr{V}}^{\alpha+1} \xrightarrow{\;\partial_t\;} \tilde{\mathscr{V}}^\alpha.$$

Working in the trivialization $\mathscr{O}_\Delta \otimes_{\mathbb{C}} V \cong \tilde{\mathscr{V}}^\alpha$ where the connection takes the form $\nabla(1 \otimes v) = \frac{dt}{t} \otimes Rv$, we get

$$\partial_t(t \otimes v) = 1 \otimes v + t \cdot \frac{1}{t} \otimes Rv = 1 \otimes (R + \mathrm{id})v.$$

Since the eigenvalues of R belong to the interval $[\alpha, \alpha+1)$, the operator $R + \mathrm{id}$ is invertible as long as $\alpha \leq -2$. This shows that $\partial_t t$ is an isomorphism, and so $\partial_t \colon \tilde{\mathscr{V}}^{\alpha+1} \to \tilde{\mathscr{V}}^\alpha$ must be surjective. □